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Effect of mobility in partially occupied complex networks

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The collective dynamics of coupled oscillators has been well studied in fully occupied networks, but little attention has been paid to the case of partially occupied networks. We study this problem by a dynamic bipartite model and focus on the influence of population mobility. We find that when the density of occupied nodes is smaller than the percolation threshold ρ_c, the order parameter will show an effect of mobility with optimal value at a medium moving probability. Its mechanism can be revealed through three factors, i.e., the size of the largest component, the mixing degree in individual components, and the frequency of exchange information among components. When the density of occupied nodes is larger than ρ_c, the moving probability will act as a bifurcation parameter to synchronization. The effect of mobility also exists for other dynamics such as the epidemic spreading where the effect is shown through the number of infected agents.

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The complex networks are ubiquitous in nature and have been intensively studied in the past decade [1–4]. It is found that the dynamics on networks can be seriously influenced by the network structure. Two important dynamics among them are the synchronization and epidemic spreading. Suppose each node is occupied by one oscillator. It is revealed that the synchronization on a network is determined by the coupling matrix, the dynamical function of oscillator, and the coupling strength [4] and can be used to explain some biological phenomena such as the hearing sensitivity on weak signals [5].

When the nodes are occupied by agents, the attention is paid on the epidemic spreading [1–3]. There are two frameworks of epidemic spreading: static and dynamic. For the former, each node is occupied by one agent and the virus/disease can be only transmitted through the links to the nearest-neighboring nodes of the infected ones. An important result is that the epidemic can spread to the entire network even when the probability of transmission is infinitely small for a scale-free (SF) network [6]. For the latter, a node can be occupied by multiple agents and the agents can move from one node to another through the links [7–11]. The used model is thus called reaction-diffusion model where the contagious process occurs only within the agents staying at the same node and the epidemic spread out by the diffusion process [9].

Although the study of the influence of network structure on dynamics has achieved great success, the considered interaction is sometimes questionable or far from reality. For example, Ref. [12] shows how to figure out a social-contact network by tracking the paths of agents. Agents may take different activities in different time intervals such as home, work, school, and social/recreational activities. Simplifying the visited locations as nodes and the paths as links, the social-contact network is obtained [12]. Thus, agents' activities on this network become a bipartite graph, which is very convenient for the study of the epidemic spreading [9–14]. However, as the links of a node do not exist at the same time but in different time periods, it is not reasonable to let all the links of a node take effect at the same time. Another example is the sex contact network where the different links of a node may come from different time periods; i.e., the old sex relationship may break up when the new sex relationship is established [15]. Thus, it is artificial to let all the links have the same probability to transmit disease at the same time. How to precisely represent the dynamics on such time-dependent links is an open question.

Furthermore, we may often observe another situation where the nodes are occupied in one time and empty in another time such as the cafeterias and conference halls in the school network. A direct evidence coming from the spreading of Bluetooth viruses through mobile phone users is studied [11]. For these situations, the nodes are partially occupied and the occupied status depends on time. It is thus interesting to know what the dynamics is on these networks. For this purpose, we here consider the case that each individual node can be occupied by at most one oscillator/agent and the density of occupied nodes is less than unity. We surprisingly find that when the density of occupied nodes is smaller than the percolation threshold ρ_c, an effect of mobility will show up with optimal value at a medium moving probability. Its mechanism can be revealed through three factors, i.e., the size of giant component, the mixing degree in components, and the frequency of exchange information among components. When the density of occupied nodes is larger than ρ_c, the moving probability will act as a bifurcation parameter to synchronization. The effect of mobility is not only for the coupled oscillators, but also for other dynamics such as the epidemic spreading.

We first construct a SF Barabasi and Albert (BA) network with degree distribution $P(k)\sim k^{-3}$ and average degree $\langle k\rangle =6$ (i.e., $m=3$ in [1]). Then we randomly choose N nodes from the constructed network with size L and let each one be occupied by one oscillator. The density of occupied nodes is $\rho=N/L<1$. We assume that at each time step, each oscillator will randomly move to one of its empty neighbors with probability p provided that there is at least one empty neighbor there. The status of all the oscillators is parallel updated. In detail, we let each oscillator be a Kuramoto phase oscillator and thus the coupled oscillators can be expressed as

$$\dot{\phi}_i = \omega_i + \varepsilon \sum_{j=1}^{k_i} a_{ij}(t)\sin(\phi_j - \phi_i), \qquad (1)$$

where $i=1,2,\ldots,L$, k_i is the degree of node i, ε is the coupling strength, ω_i is random uniformly distributed in

$[-1,1]$, $a_{ij}(t)=1$ if both the nodes i and j are connected and occupied at the same time and $a_{ij}(t)=0$ otherwise. We find that after the transient process, the collective behavior of the coupled oscillators is uncorrelated for small coupling strength and correlated for large coupling strength. To measure the correlation we introduce the order parameter

$$r(t)e^{i\psi(t)} = \frac{1}{\rho L}\sum_{j=1}^{\rho L} e^{i\phi_j(t)}, \tag{2}$$

where $\psi(t)$ is the average phase at time t and $r(t)$ represents the degree of correlation among all the ϕ_j. We have $r(t)=1$ if all the oscillators are synchronized and $r(t)=0$ if they are uncorrelated. Considering the time dependence of $r(t)$, we let

$$R = \lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T} r(t) \tag{3}$$

be the order parameter.

When the density ρ is small, the oscillators will form a number of isolated components. With the increase in ρ, the components will merge and form larger components [16]. A giant component will appear when ρ becomes larger than a threshold ρ_c, which is equivalent to a percolation problem. This merging process is independent of the concrete dynamics on the network. The value of ρ_c can be estimated as follows. Suppose the number of occupied neighbors of a node is k'. Considering that the decrease in density is equivalent to the decrease in the average $\langle k'\rangle$, the condition for the appearance of ρ_c can be figured out by the percolation of occupied bond. For a random network, Newman *et al.* showed that the condition of phase transition at which a giant component first appears is $\Sigma_{k'}k'(k'-2)P(k')=0$ [14], where $P(k')$ is the degree distribution. This condition means $\langle k'\rangle=2$. On the other hand, the existence of a giant component is equivalent to construct a BA network by a minimum m, i.e., $m=1$, which also means $\langle k'\rangle=2$. Thus, for the partially occupied network with $m>1$ we have $2mL\rho_c=2L$, which gives $\rho_c=1/m$. In this paper, we let the BA network have $L=1000$ and $\langle k\rangle=6$ (i.e., $m=3$) if without specific illustration, thus, $\rho_c=1/3$.

The synchronization for the case of $\rho=1$ has been well studied [4,16]. In this paper, we focus on the situation around ρ_c. When $\rho<\rho_c$, there is no giant component and the number of components may change with time. Suppose the N oscillators at time t are separated into M components. In each component we have

$$\dot{\phi}_i^x = \omega_i^x + \varepsilon\sum_{j=1}^{k_i} a_{ij}(t)\sin(\phi_j^x - \phi_i^x), \tag{4}$$

where $x=1,2,\ldots,M$. The $r(t)$ in Eq. (2) is an average on the components and can be rewritten as

$$r(t)e^{i\psi(t)} = \langle r^x(t)e^{i\psi^x(t)}\rangle = \frac{1}{N}\sum_{x=1}^{M} N^x r^x(t)e^{i\psi^x(t)}, \tag{5}$$

where $r^x(t)$, $\psi^x(t)$, and N^x are the order parameter, average phase, and size of component x, respectively. Therefore, the R in Eq. (3) is an average on the components. For larger M, there will be more different $\psi^x(t)$ and thus smaller $r(t)$, and vice versa, indicating that the merging of components will increase R and the decomposing will decrease R.

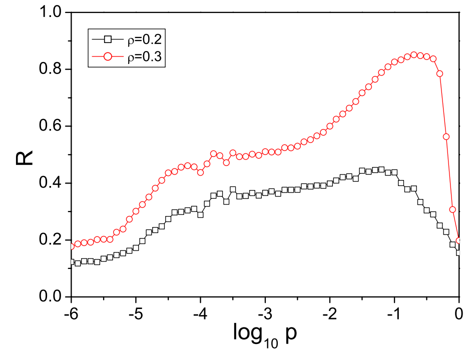

FIG. 1. (Color online) R versus p with network size $L=1000$ and average degree $\langle k\rangle=6$, where the squares and circles represent the cases of $\rho=0.2$ and 0.3, respectively.

Let us do numerical simulations as follows. After the transient process of Eq. (1), we calculate its order parameter R according to Eq. (3). We interestingly find that R will increase with p and then decrease with the further increase in p; i.e., there is an optimal p. Figure 1 shows the results for $\epsilon=0.5$ and $\rho<\rho_c$, where the "squares" and "circles" represent the cases of $\rho=0.2$ and 0.3, respectively.

What is the mechanism of the existence of optimal p? To find the answer, we divide the system variations caused by p into three parts: the varying of the average size of the largest component $S_{\max}$ or the varying of the average number of components N_{com}, the exchanging information between components, and the mixing in individual components. Let us first check the varying of $S_{\max}$ and N_{com}. We find that there is a common optimal region of p for both $S_{\max}$ and N_{com} where $S_{\max}$ shows maximum while N_{com} shows minimum. The inverse relationship between $S_{\max}$ and N_{com} is easy to be understood as the decrease in $S_{\max}$ will result in more small clusters and thus the increase in N_{com}. Figure 2 shows the results where the squares and circles represent the cases with $\rho=0.2$ and 0.3, respectively. Substituting these results into

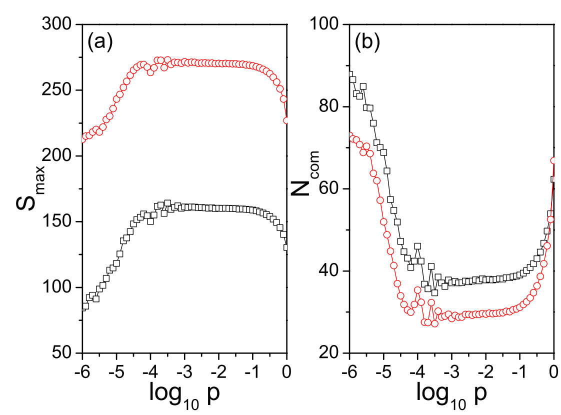

FIG. 2. (Color online) The average size of the largest component $S_{\max}$ and the average number of components N_{com} in Fig. 1, where the squares and circles represent the cases with $\rho=0.2$ and 0.3, respectively.

Eq. (5) we see that the existence of optimal $S_{\max}$ and N_{com} on p will result in the existence of optimal R on p indicating the consistence between R and $S_{\max}$.

Theoretically, we may explain the existence of optimal $S_{\max}$ on p as follows. For the case of small p, the number of moving oscillators from a component at each time step is small. Once the largest component is formed, it will be sustained for a relatively long time. Its evolution can be given as follows. At each time step, the number of moving oscillators in $S_{\max}$ will be $pS_{\max}$ and the number of nonmoving oscillators is $(1-p)S_{\max}$. Considering that the moving oscillators can only go to their neighbors at one time step, the new destinations of the $pS_{\max}$ can be classified into two parts: one part is still connected with the $(1-p)S_{\max}$ and is thus proportional to the occupied area of the $(1-p)S_{\max}$; the other is disconnected with the $(1-p)S_{\max}$ and can be expressed as $pS_{\max}[1-(1-p)S_{\max}/S_{\max}]=p^2S_{\max}$. The second part will be really lost from the component. At the same time, some small components S_i or moving oscillators from S_i may join $S_{\max}$ by a probability p_1, which is proportional to the occupied area of the $(1-p)S_{\max}$. Thus, we have

$$\frac{dS_{\max}}{dt} = -p^2S_{\max} + S_i c_1(1-p)S_{\max}, \tag{6}$$

where c_1 is a coefficient. Letting $dS_{\max}/dt=0$ and $S_i \equiv c_2S_{\max}$ with $c_2<1$, we obtain the stationary solution

$$S_{\max} = \frac{p^2}{c_1c_2(1-p)}. \tag{7}$$

From Eq. (7) one can easily see that $S_{\max}$ increases with p confirming the increasing parts in Fig. 2(a) for small p.

When p is relatively large, the moving part $pS_{\max}$ will not be a small part of $S_{\max}$ and thus may make the component $S_{\max}$ decompose into small components. In this sense, the largest component $S_{\max}$ cannot be sustained resulting in the decrease in $S_{\max}$. This is the decreasing part in Fig. 2(a) for larger p.

Then we check the effect of the exchanging information between components. Considering that information is transmitted by the moving oscillators, we design a small network to illustrate the influence of exchanging information between components, which consists of two subnetworks with the same size $L_s=100$. The connection between the two subnetworks is implemented by a moving oscillator called messenger. Thus, the total size of the network is 201. In each subnetwork, the oscillators are located on a circle with the nearest neighboring coupling by a coupling strength ε_1. We let all the oscillators satisfy Eq. (1) and let their ω_i be uniformly distributed in [0,1]. The messenger has $\omega=0.5$. When the messenger is located at one subnetwork, it will contact all the oscillators there by a coupling strength ε_2 and have a possibility p to move to the other one and vice versa. The inset of Fig. 3(a) shows its schematic figure. As the oscillators are nonidentical, the correlations in the two subnetworks will be different. The messenger will make the average phases of the two subnetworks close to each other; thus, the increase in p will increase the order parameter R. Figure 3(a) shows the result for $\varepsilon_1=0.2$ and $\varepsilon_2=0.5$.

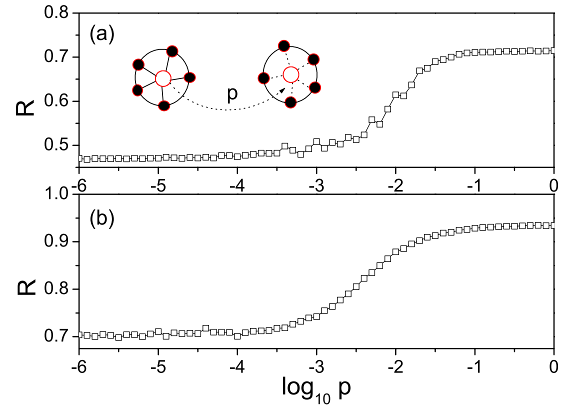

FIG. 3. (Color online) (a) How the exchanging information between two subnetworks influences R; see text for details. (b) How the mixing in a component influences R.

Finally, we check the case of mixture in individual components. Take one component as an example. We produce a small BA network with $L_s=200$ as the component and let every node be occupied by an oscillator of Eq. (1). The ω_i are uniformly distributed in $[-1,1]$ and the coupling strength is taken as $\varepsilon=0.3$. To simulate the mixing process, at each time step, we let every oscillator change its location with one of its neighbors with probability p. We find that R increases monotonously with p. Figure 3(b) shows the result. In sum, both the exchanging information between components and the mixing in individual components will increase R. Thus, the decreasing part of Fig. 1 is only from the decreasing part of $S_{\max}$ in Fig. 2(a). That is, the existence of optimal $S_{\max}$ is the main reason for the existence of optimal R on p.

Now, let us move to the case that the density of oscillators ρ is slightly larger than the threshold ρ_c, where a giant component may appear. In this situation, we find a very interesting result; i.e., the moving probability p acts as a bifurcation parameter and results in a phase synchronization when p is relatively large. Figure 4(a) shows the result for BA network with size $L_s=100$, where $\Omega_i=\langle\dot{\phi}_i\rangle$. For comparison, we also put the results of how Ω_i changes with the density ρ and coupling strength ε there. It is easy to see that p has the same function with ρ and ε. We have also made numerical simu-

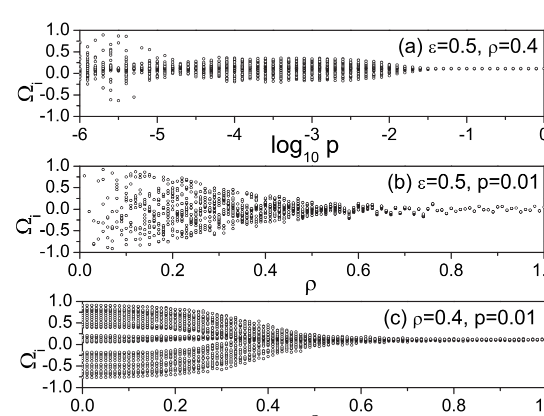

FIG. 4. (a) Ω_i versus p for $\varepsilon=0.5$ and $\rho=0.4$; (b) Ω_i versus ρ for $\varepsilon=0.5$ and $p=0.01$; (c) Ω_i versus ε for $\rho=0.4$ and $p=0.01$.

lations on random networks and found the similar resonance effect and bifurcation diagram.

We have to point out that the observed effect of mobility is not only for the phase oscillators but also works for other dynamics such as the epidemic spreading. To illustrate it, we replace Eq. (1) by the susceptible-infected-susceptible (SIS) model [6,17] to describe the infectious process. In the SIS model, nodes can be in two distinct states: healthy and ill. Suppose the susceptible has a probability λ of contagion with each infected neighbor. If the node i is susceptible, and that it has k_i neighbors, of which k_{inf} are infected, then at each time step node i will become infected with probability $[1-(1-\lambda)^{k_{inf}}]$. At the same time, each infected node will become susceptible at rate μ at each time step. To be brief, let us set $\mu=1$. Figure 5 shows the results on the BA network used in Fig. 1 where $\lambda=0.5$ and the squares and circles represent the density of oscillators $\rho=0.2$ and 0.3, respectively. Obviously, the final infected population I shows the effect of mobility, i.e., the existence of optimal I on p.

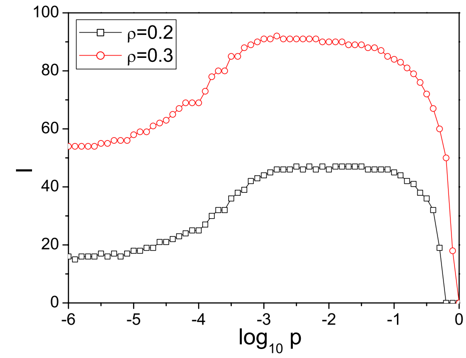

FIG. 5. (Color online) I versus p in the network used in Fig. 1.

In conclusion, we have studied the moving of oscillators in a partially occupied BA network and found an effect of mobility. This result is not only limited to the phase of oscillators but also works for the epidemic spreading. Considering that the partially occupied networks are ubiquitous in biology, social contact, and physical networks, our results may open a window to them in contrast to the previous studies on fully occupied networks.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. **424**, 175 (2006).
[3] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
[4] A. Arenas, A. Diaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. **469**, 93 (2008).
[5] J. A. Acebrón, S. Lozano, and A. Arenas, Phys. Rev. Lett. **99**, 128701 (2007); Z. Liu and T. Munakata, Phys. Rev. E **78**, 046111 (2008); F.-M. Lu and Z.-H. Liu, Chin. Phys. Lett. **26**, 040503 (2009).
[6] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001); Phys. Rev. E **63**, 066117 (2001); **65**, 035108(R) (2002).
[7] L. Hufnagel, D. Brockmann, and T. Geisel, Proc. Natl. Acad. Sci. U.S.A. **101**, 15124 (2004).
[8] A. Gautreau, A. Barrat, and M. Barthelemy, J. Stat. Mech.: Theory Exp. (2007), L09001.
[9] V. Colizza, R. Pastor-Satorras, and A. Vespignani, Nat. Phys. **3**, 276 (2007); V. Colizza and A. Vespignani, Phys. Rev. Lett. **99**, 148701 (2007); J. Theor. Biol. **251**, 450 (2008).
[10] M. Tang, L. Liu, and Z. Liu, Phys. Rev. E **79**, 016108 (2009); M. Tang, Z. Liu, and B. Li, Europhys. Lett. **87**, 18005 (2009); J. Zhou and Z. Liu, Physica A **388**, 1228 (2009).
[11] M. C. González, C. A. Hidalgo, and A.-L. Barabási, Nature (London) **453**, 779 (2008).
[12] S. Eubank, H. Guclu, V. S. A. Kumar, M. V. Marathe, A. Srinivasan, Z. Toroczkai, and N. Wang, Nature (London) **429**, 180 (2004).
[13] G. Chowell, J. M. Hyman, S. Eubank, and C. Castillo-Chavez, Phys. Rev. E **68**, 066102 (2003).
[14] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[15] F. Liljeros, C. R. Edling, L. A. N. Amaral, H. E. Stanley, and Y. Aberg, Nature (London) **411**, 907 (2001).
[16] J. Gómez-Gardeñes, Y. Moreno, and A. Arenas, Phys. Rev. E **75**, 066106 (2007); this paper reveals a similar merging process on the giant synchronized component for fully occupied networks, where the changing parameter is the coupling strength.
[17] Z. Liu and B. Hu, Europhys. Lett. **72**, 315 (2005).